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Bernstein's Inequality in L^p for $0 < p < 1$*

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One of the most powerful tools in approximation theory is Bernstein's inequality,

$$\int_{-\pi}^{\pi} |T_n'(t)|^p \, dt \leqslant C(p)\, n^p \int_{-\pi}^{\pi} |T_n(t)|^p \, dt \tag{1}$$

which holds for $1 \leqslant p \leqslant \infty$ with $C(p) = 1$. (For $p = \infty$ it should be interpreted in an obvious way.) Here T_n is an arbitrary trigonometric polynomial of order n. It has been shown recently in [3] that for $0 < p < 1$ inequality (1) still holds with some finite constant $C(p)$. Having in mind the importance of Bernstein's inequality we will give an alternative proof of (1) for $0 < p < 1$. Our method is not only simpler than that in [3] but also enables us to give a numerical estimate for $C(p)$. Using (1) we also prove weighted Markov type inequalities for algebraic polynomials.

THEOREM 1. *Let* $0 < p < 1$. Then Bernstein's inequality (1) is satisfied with $C(p) = 8p^{-1}$.

Proof. Let $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$. Then $|D_n(x)| \leqslant D_n(0) = 2n + 1$, $|D_n'(x)| \leqslant n(n+1)$ and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n^2(t)\, dt = 2n + 1.$$

If T_n is a trigonometric polynomial of order n, then convoluting T_n with D_n yields T_n, that is $T_n = T_n * D_n$. Hence $T_n' = T_n * D_n'$. Therefore we have the following two inequalities:

$$|T_n(x)| \leqslant \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} |T_n(t)|\, dt, \tag{2}$$

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$$| T_n'(x)| \leqslant \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} | T_n(t)| \, dt. \tag{3}$$

Now let $0 < p < 1$. From (2),

$$\max_{|x|\leqslant\pi} | T_n(x)| \leqslant \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt \max_{|x|\leqslant\pi} | T_n(x)|^{1-p},$$

that is,

$$| T_n(x)|^p \leqslant \frac{2n+1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt.$$

Thus by (3),

$$\begin{aligned} | T_n'(x)| &\leqslant \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt [\max_{|x|\leqslant\pi} | T_n(x)|^p]^{(1-p)/p} \\ &\leqslant \frac{n(n+1)}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt (2n+1)^{(1-p)/p} \\ &\quad \times \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt\right]^{(1-p)/p}. \end{aligned}$$

Hence

$$| T_n'(x)|^p \leqslant n^p(n+1)^p(2n+1)^{1-p} \frac{1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, dt.$$

Now let $k = [2/p] + 1$ and substitute in the last inequality $T_n(t) D_n^k(x-t)$ instead of T_n. Observe that $T_n D_n^k$ is a trigonometric polynomial of order $n(k+1)$. Therefore

$$\begin{aligned} &| T_n'(x)(2n+1)^k - k(2n+1)^{k-1} D_n'(0) T_n(x)|^p \\ &\quad \leqslant n^p(k+1)^p[n(k+1)+1]^p[2n(k+1)+1]^{1-p} \\ &\quad \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p \, | D_n(x-t)|^{kp} \, dt. \end{aligned}$$

As $D_n'(0) = 0$ and $kp \geqslant 2$, we get

$$\begin{aligned} | T_n'(x)|^p &\leqslant n^p(k+1)^p(2n+1)^{-2}[n(k+1)+1]^p[2n(k+1)+1]^{1-p} \\ &\quad \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} | T_n(t)|^p D_n^2(x-t) \, dt. \end{aligned}$$

Integrating this inequality, we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} | T_n'(x)|^p \, dx &\leqslant n^p(k+1)^p(2n+1)^{-1}[n(k+1)+1]^p[2n(k+1)+1]^{1-p} \\ &\quad \cdot \int_{-\pi}^{\pi} | T_n(t)|^p \, dt. \end{aligned}$$

Let m be a natural number. Apply the last inequality to $T_n(mx)$ instead of $T_n(x)$, divide by m^p and let $m \to \infty$. The result is

$$\int_{-\pi}^{\pi} | T_n'(x)|^p \, dx \leqslant (k+1)^{1+p} 2^{-p} n^p \int_{-\pi}^{\pi} | T_n(t)|^p \, dt.$$

Recall that $k \leqslant 2p^{-1} + 1$. The theorem follows.

It would be of definite interest to determine the best value of $C(p)$.

We now establish weihgted Markov inequalities for algebraic polynomials. Denote by $p_n(\alpha, \beta, x) = \gamma_n(\alpha, \beta) x^n + \cdots$ $(\alpha > -1,\ \beta > -1)$, the orthonormalized Jacobi polynomials, and let

$$K_n(\alpha, \beta, x) = \sum_{j=0}^{n-1} p_j^2(\alpha, \beta, x).$$

LEMMA 2. *Let* $\alpha > -1$, $\beta > -1$, $\gamma > -1$, *nonnegative integers* k, l, m, *positive integer* n, *and* $0 < \epsilon < 1$ *be fixed. Set*

$$P(x) = n^{-2} x^k (1-x)^l (1+x)^m K_n(\alpha, \beta, x) K_n(-\tfrac{1}{2}, \gamma, 2x^2 - 1). \tag{4}$$

Then

$$| P'(x)| \leqslant C_1 | x |^{-1} (1-x^2)^{-1} | P(x)| \qquad \text{for} \quad | x | \leqslant 1, \tag{5}$$

$$0 < C_2 \leqslant | P(x)| \, | x |^{-k+2\gamma+1} (1-x)^{-l+\alpha+1/2} (1+x)^{-m-\beta+1/2} \leqslant C_3 < \infty \tag{6}$$

for $\epsilon n^{-1} \leqslant | x | \leqslant 1 - \epsilon n^{-2}$, *where* C_1, C_2, *and* C_3 *are independent of* x *and* n.

Proof. First let us calculate $K_n'(\alpha, \beta, x)$. By the Christoffel–Darboux formula,

$$K_n(\alpha, \beta, x) = \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} [p_n'(\alpha, \beta, x) p_{n-1}(\alpha, B, x) - p_{n-1}'(\alpha, \beta, x) p_n(\alpha, \beta, x)].$$

Hence,

$$K_n'(\alpha, \beta, x) = \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} [p_n''(\alpha, \beta, x) p_{n-1}(\alpha, \beta, x) - p_{n-1}''(\alpha, \beta, x) p_n(\alpha, \beta, x)].$$

Note that $p_n(\alpha, \beta, x)$ satisfies the differential equation

$$(1-x^2) Y'' = -n(n+\alpha+\beta+1) Y + [\alpha - \beta + (\alpha+\beta+2)x] Y'.$$

Therefore,

$$K_n'(\alpha, \beta, x) = \frac{\alpha - \beta + (\alpha+\beta+2)x}{1-x^2} K_n(\alpha, \beta, x) - \frac{\gamma_{n-1}(\alpha, \beta)}{\gamma_n(\alpha, \beta)} \frac{2n+\alpha+\beta}{1-x^2} p_{n-1}(\alpha, \beta, x) p_n(\alpha, \beta, x).$$

It has been shown in [2] that

$$n\,|\,p_{n-1}(\alpha, \beta, x)\,p_n(\alpha, \beta, x)| \leqslant \text{const}\, K_n(\alpha, \beta, x)$$

for $|\,x\,| \leqslant 1$. Thus

$$|\,K_n'(\alpha, \beta, x)| \leqslant \text{const}(1 - x^2)^{-1}\, K_n(\alpha, \beta, x)$$

for $|\,x\,| \leqslant 1$, which yields (5) by a simple computation. Concerning (6), see, e.g., [2, Section 6.3].

LEMMA 3 [2, Section 6.3]. *Let* $\alpha > -1$, $\beta > -1$, $\gamma > -1$, *and* $0 < p < \infty$. *Then there exists* $\delta > 0$ *such that, for every algebraic polynomial* π_n *of degree at most* n,

$$\int_{-1}^{1} |\,\pi_n(t)|^p(1 - t)^\alpha(1 + t)^\beta\,|\,t\,|^\gamma\, dt \leqslant 2 \int_{\delta/n \leqslant |t| \leqslant 1-\delta/n^2} |\,\pi_n(t)|^p(1 - t)^\alpha(1 + t)^\beta\,|\,t\,|^\gamma\, dt.$$

LEMMA 4. *Let* $0 < p < \infty$, $0 < \epsilon < 1$. *Let* a, b, *and* c *be given real numbers. Then there exist constants* $\delta > 0$ *and* C_4 *such that, for every algebraic polynomial* π_n *of degree at most* n,

$$\int_{\epsilon/n \leqslant |t| \leqslant 1-\epsilon/n^2} |\,\pi_n'(t)(1 - t^2)^{1/2}\,|^p(1 - t)^a(1 + t)^b\,|\,t\,|^c\, dt \leqslant C_4 n^p \int_{\delta/n \leqslant |t| \leqslant 1-\delta/n^2} |\,\pi_n(t)|^p(1 - t)^a(1 + t)^b\,|\,t\,|^c\, dt.$$

Proof. If $a = b = -\frac{1}{2}$ and $c = 0$, then the lemma follows from Bernstein's inequality ($1 \leqslant p < \infty$), Theorem 1 ($0 < p < 1$), and Lemma 3. Otherwise we choose $\alpha, \beta, \gamma, k, l$, and m so that they satisfy the conditions of Lemma 2, and $a = p(l - \alpha - \frac{1}{2}) - \frac{1}{2}$, $b = p(m - \beta - \frac{1}{2}) - \frac{1}{2}$, and $c = p(k - 2\gamma - 1)$. Let P be defined by (4). Then $P\pi_n$ is of degree $5n + k + l + m = O(n)$. Applying the case $a = b = -\frac{1}{2}$, $c = 0$ to $P\pi_n$ instead of π_n, we easily obtain the lemma.

Lemmas 3 and 4 combined imply

THEOREM 5. *Let* $0 < p < \infty$. *Let* $\Gamma_1, \ldots, \Gamma_N$ *be arbitrary reals, and let* $1 = x_1 > x_2 > \cdots > x_N = -1$, $\gamma_i > -1$. *Set*

$$W(t) = \prod_{i=1}^{N} |\,t - x_i\,|^{\gamma_i},$$

$$W_n(t) = \left((1 + t)^{1/2} + \frac{1}{n}\right)^{2\Gamma_1} \prod_{i=2}^{N-1} \left(|\,t - x_i\,| + \frac{1}{n}\right)^{\Gamma_i} \left((1 + t)^{1/2} + \frac{1}{n}\right)^{2\Gamma_N}.$$

Then, for every algebraic polynomial π_n of degree at most n,

$$\int_{-1}^{1} |\pi_n'(t)(1-t^2)^{1/2}|^p W_n(t) W(t)\, dt \leqslant C_5 n^p \int_{-1}^{1} |\pi_n(t)|^p W_n(t) W(t)\, dt.$$

where C_5 is independent of n.

Let us remark that Theorem 5 is new only for $0 < p < 1$. For $1 \leqslant p < \infty$ it was proved in [2], but the present proof is much simpler than that in [2]. There is an extensive literature dealing with the case $N = 2$, that is, when W is a Jacobi weight. We refer the reader to [1] where many papers on weighted Bernstein inequalities are cited.

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